Light forces on dielectric particles

Markus Selmke, markus.selmke@gmx.de

March 25, 2014

Since the pioneering work of A. Ashkin [1] it has been common to express the radiation forces on small particles as the sum of two [1, 2] and sometimes as the sum of three terms [3, 4], i.e. the gradient, scattering and absorption force [3] (The last two terms combined are often referred to simply as the "scattering term"):

$$\mathbf{F} = \frac{\alpha n}{2c}\nabla I\left(\mathbf{r}\right) + \frac{n}{c}I\sigma^R_{\mathrm{sca}}\hat{\mathbf{k}} + \frac{n}{c}I\sigma^R_{\mathrm{abs}}\hat{\mathbf{k}} = \mathbf{F}_{\mathrm{gradient}} + \mathbf{F}_{\mathrm{sca}} + \mathbf{F}_{\mathrm{abs}} \tag{1}$$

with the RAYLEIGH scattering and absorption cross-sections $\sigma^R_{\mathrm{sca}} = k^4|\alpha|^2/6\pi$ and $\sigma^R_{\mathrm{abs}} = k\Im\left(\alpha\right)$, respectively. The extinction cross-section is the sum of both, $\sigma_{\mathrm{ext}} = \sigma_{\mathrm{abs}} + \sigma_{\mathrm{sca}}$. For the gradient term, originally used for real-valued polarizabilities α only, some articles use $\Re\left(\alpha\right)$ while others use $|\alpha|$. Incorrect expressions can be found on wikipedia `http://en.wikipedia.org/wiki/Optical_tweezers` and in the classic reference book of A. Ashkin [5, p. 100, 458], see Fig. 1.

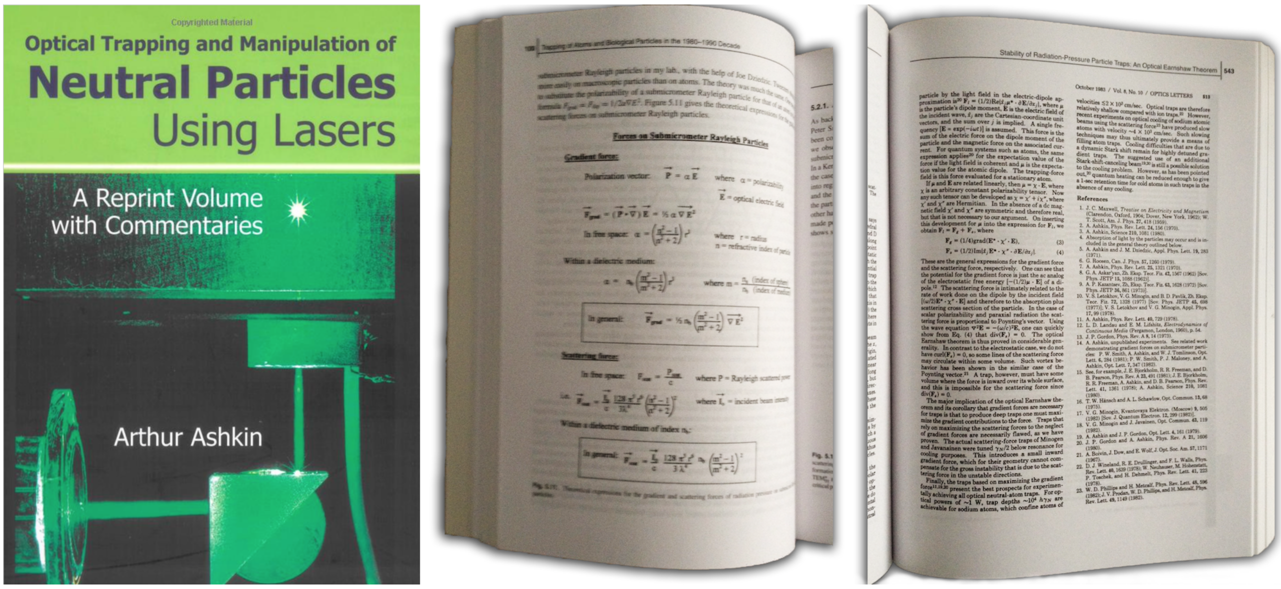

Figure 1: **Ashkin's book:**[5] Often cited formulas for radiation pressure on spherical particles in a liquid. The gradient force reads $(1/2)n\alpha|\nabla\mathbf{E}|^2$ instead of $(1/4)\epsilon_0 n^2\Re\left(\alpha\right)|\nabla\mathbf{E}|^2$. Note the correct expression is also found as an attached reprint of Ashkins paper [1] on page 543.

This short note will give a brief review of the current state of affairs, showing that the above commonly used expression is incomplete. Instead, the time-averaged force is [6, 7, 8, 10, 11, 12, 13] given by Eq. (2), in principle already given by A. Ashkin in 1983 [1].

$$\left\langle\mathbf{F}\right\rangle = \frac{n\Re\left(\alpha\right)}{2c}\nabla I\left(\mathbf{r}\right) + nk\Im\left(\alpha\right)\left[\frac{\left\langle\mathbf{S}\left(\mathbf{r}\right)\right\rangle}{c} + \frac{\epsilon_0}{4k_0 i}\nabla\times\left(\mathbf{E_i}\left(\mathbf{r}\right)\times\mathbf{E_i}^*\left(\mathbf{r}\right)\right)\right] \tag{2}$$

with the radiation-back-reaction corrected polarisability $\alpha = \alpha_{\mathrm{CM}}/\left[1 - ik^3\alpha_{\mathrm{CM}}/6\pi\right]$, wave vector $k = nk_0 = n2\pi/\lambda$, c the vacuum speed of light and the zero-frequency CLAUSIUS-MOSSOTTI relation [14, p. 206] for the polarisability $\alpha_{\mathrm{CM}} = 4\pi R^3 \left[n_p^2 - n^2\right] / \left[n_p^2 + 2n^2\right]$. The prefactor $nk\Im(\alpha)$ may be approximated by $n\,\sigma_{\mathrm{ext}}^R(\alpha_{\mathrm{CM}})$ (see comment after Eq. (7)). The intensity is related to the electric field amplitude (peak) via $I = c\epsilon_0 n|\mathbf{E}|^2/2$. The POYNTING vector is $\mathbf{E}\times\mathbf{H}$, and its time-average $\langle\mathbf{S}\rangle = \Re(\mathbf{E}\times\mathbf{H}^*)/2 = |\mathbf{E}|^2 c\epsilon_0 n\hat{\mathbf{k}}/2 = I\,\hat{\mathbf{k}}$ for a plane wave.

The first term is the gradient force. The second term in the above expression includes both the scattering and the absorption contribution, while the third unnamed term relates to a polarisation gradient or the time-averaged spin density of a transverse electromagnetic field $\langle\mathbf{L_s}\rangle = (\epsilon_0 n^2/4\omega i)(\mathbf{E}\times\mathbf{E}^*)$. The last term is zero for linearly polarised light [6].

At least three analytical approaches are able provide the radiation pressure on small particles:

1 Body force density

The instantaneous Lorenzian force density in an isotropic medium reads [14, Sec. 2.6 and 2.29]:

$$\mathbf{f} = \mathbf{f}_{\mathrm{stat.}} + \mathbf{f}_{\mathrm{Abraham}} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B} + \frac{n^2-1}{c^2}\frac{\partial\mathbf{S}}{\partial t} \tag{3}$$

Arthur Ashkin and James P. Gordon found [1] a light force of $\langle\mathbf{F}\rangle = (1/2)\Re(\hat{\mathbf{e}}_{\mathbf{i}}\alpha^*\mathbf{E}^*\cdot\partial_i\mathbf{E})$. 20 years later, the same expression was rediscovered by *Patrick C. Chaumet*, starting from the body force expression Eq. (3) and correctly carrying out the time-averaging [15] (details on the derivation also in Ref. [13]). He finally arrives (with $\mathbf{E}$ denoting the incident field, and $\mathbf{p} = \epsilon\alpha\mathbf{E}$, $\epsilon = \epsilon_0 n^2$, [14, p. 206]) at Eq. (4):

$$\langle\mathbf{F}\rangle = \frac{1}{2}\Re\left(\sum_j \epsilon\alpha E_j\nabla(E_j)^*\right) \tag{4}$$

$$= \frac{n^2\epsilon_0}{2}\left\{\Re(\alpha)\frac{\nabla|\mathbf{E}|^2}{2} - k_0\Im(\alpha)\left[\frac{-1}{2ik_0}\nabla\times(\mathbf{E}\times\mathbf{E}^*) + \frac{1}{k_0}\Im(-i\omega\mu\mathbf{E}\times\mathbf{H}^*)\right]\right\} \tag{5}$$

$$= \frac{n^2\epsilon_0}{4}\Re(\alpha)\nabla|\mathbf{E}|^2 + nk\Im(\alpha)\left[\frac{c\epsilon_0}{4i\omega}\nabla\times(\mathbf{E}\times\mathbf{E}^*) + \frac{1}{2c}\Re(\mathbf{E}\times\mathbf{H}^*)\right] \tag{6}$$

The following identities were used [6]: $\Re(zw) = \Re(z)\Re(w) - \Im(z)\Im(w)$, $\Im(iz) = \Re(z)$, $\sum_j E_j\nabla(E_j)^* = (\mathbf{E}\cdot\nabla)\mathbf{E}^* + \mathbf{E}\times(\nabla\times\mathbf{E}^*)$, $2\Re((\mathbf{E}\cdot\nabla)\mathbf{E}^*) = \nabla(\mathbf{E}\cdot\mathbf{E}^*) - 2\Re(\mathbf{E}\times(\nabla\times\mathbf{E}^*))$, $2i\Im((\mathbf{E}\cdot\nabla)\mathbf{E}^*) = \nabla\times(\mathbf{E}^*\times\mathbf{E})$ (the last one only valid for $\nabla\cdot\mathbf{E} = 0$, i.e. the external field. Typo in [6]) and $\nabla\times\mathbf{E} = -\partial_t\mathbf{B} = i\omega\mu\mathbf{H}$, $\mu = \mu_0 = 1/\epsilon_0 c^2$ and $\omega = ck_0$. This is the correct expression Eq. (2). As noted in the beginning, it includes both the pressure and gradient forces.

A further interpretation, especially of the last term as a polarisation gradient force, has been given in Ref. [6]. See also the Review article Ref. [16] and Ref. [9, 11, 12, 8].

The polarisability α should be corrected by the dipoles self-field. The reason is that the dipole $\mathbf{p}$ must be finite of extent. Therefore the retarded potentials can be evaluated and expanded at small distances up to order $(r/c)^2$, i.e. for distances $r \ll \lambda$ [17]. In SI units, this radiation reaction self-field is

$$\mathbf{E_s} = \frac{ik^3\mathbf{p}}{6\pi\epsilon_0}, \quad \alpha = \frac{\alpha_{\mathrm{CM}}}{1 - ik^3\alpha_{\mathrm{CM}}/6\pi}, \quad \alpha_{\mathrm{CM}} = 4\pi R^3\frac{n_p^2 - n^2}{n_p^2 + 2n^2} \tag{7}$$

Where the last expression is the radiation corrected polarizability, first introduced by Draine in 1988, see also the comment by P.C. Chaumet [18] (the sign in the denominator depends on the time-dependence chosen!).

Expanding the radiation back-reaction corrected polarizability $\alpha \approx \alpha_{\mathrm{CM}} + ik^3|\alpha_{\mathrm{CM}}|^2/6\pi$ one finds $k\Im(\alpha) \rightarrow k\Im(\alpha_{\mathrm{CM}}) + k^4|\alpha_{\mathrm{CM}}|^2/6\pi = \sigma^R_{\mathrm{abs}}(\alpha_{\mathrm{CM}}) + \sigma^R_{\mathrm{sca}}(\alpha_{\mathrm{CM}}) = \sigma^R_{\mathrm{ext}}(\alpha_{\mathrm{CM}})$.

Alexander Rohrbach and Ernst H.K. Stelzer attempted to derive in Ref. [19] three constituting terms of the radiation pressure (extinction / absorption, scattering and gradient) through the body force which included the Abraham force term [25, 23] [14, Sec. 2.29]. The tried to establish the result (which may be expressed in terms of intensity via $I = c\epsilon_0 n|\mathbf{E^i}|^2/2$, with $\mathbf{E^i}$ being the complex amplitude of the incidence electric field)

$$\langle \mathbf{F} \rangle = \frac{\alpha n}{2cV} \iiint_V \nabla I(\mathbf{r})\, \mathrm{d}V + \frac{nI_0}{c}\left[\sigma_{\mathrm{ext}}\langle \mathbf{k_i}\rangle - \sigma_{\mathrm{sca}}\langle \mathbf{k_i}\rangle\right] \tag{8}$$

However, the expression is invalid. In their first derivation attempt the scattering forces are derived from the Abraham force term, which however is zero for optical frequencies as it averages out. A series of comments in the Journal "Applied Physics" followed:

- P. C. Chaumet: Comment on "Trapping force, force constant, and potential depths for dielectric spheres in the presence of spherical aberrations" [18]
- A. Rohrbach et al.: Reply to comment on "Trapping force, force constant, and potential depths for dielectric spheres in the presence of spherical aberrations" [21]

The last published reply by A. Rohrbach and colleagues concludes that his expressions remain correct (providing an alternative derivation), and A. Rohrbach continues to use this result with the new derivation in Ref. [22]. However, the final derivation is again based on flawed reasoning. Starting from an expression for the body force derived from Eq. (3) he writes (using a variant of the Gauss theorem)

$$\langle \mathbf{F} \rangle = \frac{1}{4V}\Re\left(\iiint_V \alpha\epsilon\nabla|\mathbf{E}|^2 \mathrm{d}V\right) = \frac{1}{4V}\Re\left(\iint_{\partial V} \alpha\epsilon\nabla|\mathbf{E}|^2\, \hat{\mathbf{e}}_{\mathbf{r}}\, \mathrm{d}A\right) \tag{9}$$

He then continues to set the field to be the sum of the incident and the scatting field, $\mathbf{E} = \mathbf{E^i} + \mathbf{E^s}$. If anything, it should be the internal field $\mathbf{E^{sp}}$ within the spherical volume. The scattered field exists only outside the scatterer. Those two fields are connected at the boundary via the electromagnetic continuity relations, i.e. the continuity of the tangential components only $\hat{\mathbf{e}}_\theta \cdot \mathbf{E^{sp}} = \hat{\mathbf{e}}_\theta \cdot \left[\mathbf{E^i} + \mathbf{E^s}\right]$ and $\hat{\mathbf{e}}_\phi \cdot \mathbf{E^{sp}} = \hat{\mathbf{e}}_\phi \cdot \left[\mathbf{E^i} + \mathbf{E^s}\right]$ (similar expressions for the magnetic field $\mathbf{H}$). However, it is instead the external field in the dipole (Rayleigh) approximation which should be considered here, see the work of P.C. Chaumet et al. [15]. Using the wrong field, he nonetheless arrives at an expression which resembles the momentum-balance. However, instead of two he finds three constituents, including $\langle\cos(\theta)\rangle\sigma_{\mathrm{sca}}$ and $\langle\cos(\theta)\rangle\sigma_{\mathrm{ext}}$. As shown by G. Gouesbet and coworkers (see section 2), the averages must be done accordingly and lead to the notion of asymmetry parameters. For these, however, the integration area must be at infinite distance instead of at the surface of the particle, as the above derivation suggests. The additional gradient term is also incorrect here as it is already included in the extinction and scattering terms.

2 (Pseudo-) Momentum Balance

Since light carries a momentum, any interaction of a beam with an arbitrary sized particle will lead to a propagation direction reconfiguration and thereby in accord with the conservation of the total linear momentum to a force exerted on the particle. This matter is far from trivial as testified by a century-old Abraham-Minkowski controversy. However, it may now be regarded as solved [25, 26] and the problem was due to a common neglect or ill-accounting for the dielectric materials stress-tensor (see also §16 in the Book of Landau and Lifschitz [23], Section 2.6 in the book of Stratton [14]). The distribution of stresses and momenta among the electromagnetic field and the medium is largely arbitrary and both methods are equivalent once the correct material counterparts are included in any specific computation. However, the MINKOWSKI tensor may usually be used without this complication, as the materials part is then typically negligible.

Thus, the MINKOWSKI momentum density for light $\mathbf{p}_{\mathrm{M}} = \frac{c}{n}\mathbf{D}\times\mathbf{B} = \frac{n}{c}\mathbf{S}$, and not the ABRAHAM momentum density $\mathbf{p}_{\mathrm{A}} = \frac{c}{n}\epsilon_0\mu_0\mathbf{E}\times\mathbf{H} = \frac{1}{cn}\mathbf{S}$ should be used for force computations on particles. It is the canonical part of the momentum of light which is responsible for the caused motion of dielectric particles in a medium and thereby the origin of the force exerted on the particle [26, 25, 24]. The force then reads [30]:

$$\langle\mathbf{F}\rangle = -\frac{n}{c}\frac{1}{2}\Re\left(\lim_{r\to\infty}\iint_{4\pi}\left(\mathbf{S}^{\mathbf{t}} - \mathbf{S}^{\mathbf{i}}\right)\mathrm{d}A\right), \tag{10}$$

with c being the speed of light in vacuum and n the refractive index of the embedding medium. The momentum flux of the incidence beam is zero.

In 1985, *Gérard Gouesbet* developed the generalised LORENZ-MIE theory (GLMT) [28]. In a later article, they gave the correct expressions for the radiation pressure in an arbitrary beam. He expresses ([27, p. 1435, (133)]) the force via radiation pressure cross-sections:

$$\sigma_{\mathbf{pr}} = c\,\mathbf{F}/I_0 = (\overline{\cos})\,\sigma_{\mathrm{ext}} - (\overline{\cos})\,\sigma_{\mathrm{sca}}, \tag{11}$$

With $I_0 = |\mathbf{E}|^2\sqrt{\epsilon/\mu}/2 = c\epsilon_0 n|\mathbf{E}|^2/2$ usually set to 1 in the GLMT theory. These two terms include all aspects of the radiation force, i.e. embody scattering, absorption and gradient force components all at the same time. Explicitly, the force is evaluated via:

$$\langle\mathbf{F}\rangle = \frac{1}{c}\iint_{4\pi}\frac{1}{2}\mathfrak{Re}\left(\mathbf{E}^{\mathbf{t}}\times\mathbf{H}^{\mathbf{t}*}\right)_r\begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix}\mathrm{d}A, \tag{12}$$

with the axial pressure cross-sections for instance being:

$$(\overline{\cos})\,\sigma_{\mathrm{ext}} = \int_0^{\pi}\int_0^{2\pi}\frac{1}{2}\Re\left(E^i_\phi H^{s*}_\theta + E^s_\phi H^{i*}_\theta - E^i_\theta H^{s*}_\phi - E^s_\theta H^{i*}_\phi\right)\cos(\theta)\sin(\theta)\,r^2\mathrm{d}\theta\mathrm{d}\phi \tag{13}$$

$$(\overline{\cos})\,\sigma_{\mathrm{sca}} = \int_0^{\pi}\int_0^{2\pi}\frac{1}{2}\Re\left(E^s_\theta H^{s*}_\phi - E^s_\phi H^{s*}_\theta\right)\cos(\theta)\sin(\theta)\,r^2\mathrm{d}\theta\mathrm{d}\phi. \tag{14}$$

Apart from the factor $\cos(\theta)$ in the integrand, they are identical to the expressions for the cross-sections σ_{ext} and σ_{sca}. It appears that a multiplication by a factor of n is necessary to make these formulas correct.

There is no reason a-priori which admits the averaging of the cosine to be done independently. Indeed, G. Gouesbet and coworkers correctly evaluate these averages together with the fields. This is an important point that occasionally leads to incorrect derivations (see the note on the work of A. Rohrbach in section 1). Only in special circumstance can this be avoided [27]:

Brian Stout also provides an analysis of the individual contributions to the forces using this approach, see Sec. 3B and C of Ref. [29]. Starting from the time-averaged stress-tensor $T_{ij} = (1/2)\,\Re\left(\epsilon\,E_i^*E_j + B_i^*B_j/\mu - (\epsilon|\mathbf{E}|^2 + |\mathbf{B}|^2/\mu)\delta_{ij}/2\right)$ he arrives at 2 contributions for the force $\mathbf{F}\propto k^2\sigma_{\mathrm{pr}}/\pi$, with $\sigma_{\mathbf{pr}} = \sigma_{\mathbf{a}} + \sigma_{\mathbf{r}}$. The first contribution is due to asymmetric scattering

In our previous paper[41] the first term was set equal to C_{ext}. This was certainly a good approximation, since only a waist location of the particle was considered; i.e., we assumed that the wave front on the scatterer was (nearly) a plane, which permitted us to use the same formulation as given by van de Hulst.[55] In the present paper, in which arbitrary location of the particle is considered, this assumption must be relaxed.

of the particle, and the second due to the momentum-flux removed from the incident beam

$$\sigma_{\mathbf{a}} = \lim_{r\to\infty} \frac{1}{2I(\mathbf{0})} r^2 \left(\frac{\epsilon}{\mu}\right)^{1/2} \iint_\Omega \hat{\mathbf{e}}_{\mathbf{r}} \left(\mathbf{E_s}^* \cdot \mathbf{E_s}\right) \mathrm{d}\Omega, \tag{15}$$

$$\sigma_{\mathbf{r}} = -\lim_{r\to\infty} \frac{1}{2I(\mathbf{0})} r^2 \left(\frac{\epsilon}{\mu}\right)^{1/2} \iint_\Omega \hat{\mathbf{e}}_{\mathbf{r}}\, \Re\left(\mathbf{E_s}^* \cdot \mathbf{E_i} + \mathbf{E_i}^* \cdot \mathbf{E_s}\right) \mathrm{d}\Omega \tag{16}$$

Continuing to assume an axially symmetric beam relative to a particle (on-axis situation), the article continues to show that the above expression $\sigma_{\mathbf{pr}} \cdot \hat{\mathbf{e}}_{\mathbf{z}}$ results in two forces corresponding to the DEBYE-MIE result $\sigma_z = \sigma_{\text{ext}} - g\sigma_{\text{sca}}$ only for plane waves. Herein $g = \hat{\mathbf{e}}_{\mathbf{z}} \cdot \sigma_{\mathbf{a}}/\sigma_{\text{sca}}$ is the usual DEBYE-MIE asymmetry parameter. Otherwise the terms can not be simplified any further in this formalism any further! Noting that $(\epsilon/\mu)^{1/2} = c\epsilon_0 n$, the above description is equivalent to the GLMT results.

3 Maxwell stress tensor (J. P. Barton's ABT)

In 1988 *John P. Barton* introduced the arbitrary beam theory. He uses the MINKOWSKI form of the MAXWELL-stress-tensor, $T_{ij} = \epsilon E_i E_j + H_i H_j/\mu - (\epsilon|\mathbf{E}|^2 + |\mathbf{H}|^2/\mu)\delta_{ij}/2$. The force on the particle then may be calculated by integrating the stress over a sphere containing the particle via

$$\langle \mathbf{F} \rangle = \langle \oint_{4\pi} \hat{\mathbf{e}}_{\mathbf{r}} \cdot \overleftrightarrow{\mathbf{T}} \rangle \tag{17}$$

Gouesbet et al. claim in their book on the GLMT that their formalism yields equivalent results up to a normalising factor (probably n?):

> The approach used in this section to evaluate radiation pressure forces would be qualified as being heuristic by Bohren and Huffman ([BH83], p 120). A more rigorous approach would rely on the use of the energy-momentum tensor. Such an approach has been developed by Barton et al [BAS89]. After a bit of algebra, it can be checked that both approaches lead to equivalent results, within irrelevant normation prefactors.

Patrick C. Chaumet used the MAXWELL-stress-tensor in combination with the the fields of a dipole, i.e. employing the dipole-approximation, to show that again the correct expression Eq. (4) may be obtained [32]. He also includes the contributions due to magnetic moments in this calculation.

Based on the GLMT, *Alessandro Salandrino* finally approached the task of cutting down the exact analytic formulas to a manageable and interpretable result for small scatterers (retaining only the scattered field's electric dipole contribution, given for a particle in free space) [30]. After an exhausting amount of algebra he finally arrived at a time-averaged force of

$$\langle \mathbf{F} \rangle = \frac{\epsilon_0}{4} \Re(\alpha)\, \nabla |\mathbf{E_i}|^2 \Big|_{r=0} + k_0 \Im(\alpha) \left[\frac{\Re\left(\mathbf{E_i} \times \mathbf{H_i}^*\right)}{2c} + \frac{\epsilon_0}{4k_0 i} \nabla \times \left(\mathbf{E_i} \times \mathbf{E_i}^*\right) \right] \tag{18}$$

A. Salandrino used a field expansion similar to the GLMT theory formalism used by G. Gouesbet but correctly considered the Minkowski momentum tensor, i.e. $n\mathbf{S}/c$ instead of $\mathbf{S}/c$ for the momentum (although he finally considers a RAYLEIGH particle in vacuum only).

References

[1] A. Ashkin and J.P. Gordon, "Stability of radiation-pressure particle traps: an optical Earnshaw theorem", Optics Lett. **8**(10) 511–513 (1983).

[2] Y. Harada et al., "Radiation forces on a dielectric sphere in the Rayleigh scattering regime", Opt. Commun. **124**, 529–541 (1996).

[3] K. Svoboda et al., "Optical trapping of metallic Rayleigh particles", Opt. Lett. **19**(13), 930-932 (1994).

[4] P. M. Hansen et al., "Expanding the Optical Trapping Range of Gold Nanoparticles", Nano Lett. **5**(10) 1937–1942 (2005).

[5] A. Ashkin, "Optical Trapping and Manipulation of Neutral Particles Using Lasers", World Scientific Publishing Company, Incorporated, 2006

[6] S. Albaladejo et al., "Scattering Forces from the Curl of the Spin Angular Momentum of a Light Field", Phys. Rev. Lett. **102** 113602 (2009).

[7] Silvia Albaladejo et al., "Giant Enhanced Diffusion of Gold Nanoparticles in Optical Vortex Fields", Nano Lett. **9**(10), 3527–3531 (2009).

[8] Raquel Gómez-Medina et al., "Nonconservative electric and magnetic optical forces on submicron dielectric particles", Phys. Rev. A **83**, 033825 (2011).

[9] V. Wong and M. A. Ratn, "Gradient and nongradient contributions to plasmon-enhanced optical forces on silver nanoparticles", Phys. Rev. B **73** 075416 (2006).

[10] I. Iglesias and Juan J. Senz, "Light spin forces in optical traps: comment on 'Trapping metallic Rayleigh particles with radial polarization' ", Opt. Express **20**(3) 2832 (2012).

[11] A. Canaguier-Durand et al., "Force and torque on an electric dipole by spinning light fields", Phys. Rev. A **88**, 033831 (2013).

[12] I. Liberal et al., "Electromagnetic force density in electrically and magnetically polarizable media", Phys. Rev. A **88**, 053808 (2013).

[13] M. V. Berry et al., "Physical curl forces: dipole dynamics near optical vortices", J. Phys. A: Math. Theor. **46** 422001 (2013).

[14] J. A. Stratton, "Electromagnetic Theory", `http://archive.org/stream/electromagnetict031016mbp#page/n223/mode/2up`

[15] P.C. Chaumet et al., "Time-averaged total force on a dipolar sphere in an electromagnetic field", Optics Lett., **25**(15), 1065ff, (2000).

[16] O. M. Maragò at al., "Optical trapping and manipulation of nanostructures", Nature Nanotechnology **8**, 807–819 (2013).

[17] N. Itoh, "Radiation reaction due to magnetic Dipole radiation", Phys. Rev. A **43**(2) (1991).

[18] P. C. Chaumet, "Comment on 'Trapping force, force constant, and potential depths for dielectric spheres in the presence of spherical aberrations' ", Appl. Opt. **43**(9) 1825–1826 (2004).

[19] A. Rohrbach et al., "Optical Trapping of dielectric particles in arbitrary fields", J. Opt. Soc. Am. A. **18**(4), p. 839ff., (2001).

[20] A. Rohrbach et al., "Trapping forces, force constants, and potential depths for dielectric spheres in the presence of spherical aberrations", Appl. Optics **41**(13), 2494ff. (2002).

[21] A. Rohrbach et al., "Reply to Comment on 'Trapping force, force constant, and potential depths for dielectric spheres in the presence of spherical aberrations' ", Appl. Opt. **43**(9) 1827–1829 (2004).

[22] A. Rohrbach, "Stiffness of Optical Traps: Quantitative Agreement between Experiment and Electromagnetic Theory", Phys. Rev. Lett. **95** 168102 (2005).

[23] L. Landau and L. Lifschitz, *Lehrbuch der theoretischen Physik*, Band VIII (8), Elektrodynamik der Kontinua", Akademie-Verlag Berlin (1990).

[24] S. M. Barnett, "Resolution of the Abraham-Minkowski Dilemma", Phys. Rev. Lett. **104**, 070401 (2010).

[25] R. N. C. Pfeifer et al., "Colloquium: Momentum of an electromagnetic wave in dielectric media", Rev. Mod. Phys. **79**(4), 1197-1216 (2007).

[26] J. P. Gordon, "Radiation Forces and Momenta in Dielectric Media", Phys. Rev. A **8**(1), 14–21 (1973).

[27] G. Gouesbet et al. "Light scattering from a sphere arbitrarily located in a Gaussian beam, using a Bromwich formulation", J. Opt. Soc. Am. A **5**(9), 1427–1443 (1988).

[28] G. Gouesbet et al., "Scattering of a Gaussian beam by a Mie scattering center using a Bromwich Formalsim", J. Optics (Paris) **16**(2), 83-93 (1985).

[29] O. Moine and B. Stout, "Optical force calculations in arbitrary beams by use of the vector addition theorem", J. Opt. Soc. Am. B **22**(8), 1620–1631 (2005).

[30] A. Salandrino, "Generalized Mie theory of optical forces", J. Opt. Soc. Am. B **29**(4) 855–866 (2012).

[31] J. P. Barton et al., "Theoretical determination of net radiation force and torque for a spherical particle illuminated by a focused laser beam", J. Appl. Phys. **66**(10), 4594 (1989).

[32] P. C. Chaumet et al., "Electromagnetic force and torque on magnetic and negative-index scatterers", Opt. Express **17**(4), 2224–2234 (2009).